

# Integrable mappings derived from the $\Delta\Delta RsG$ equation

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## Abstract

In this paper we consider the integrability of the mappings derived from the double discrete related sine-Gordon ( $\Delta\Delta RsG$ ) equation, which we recently introduced in the paper *Higher dimensional integrable mappings*, under an appropriate periodicity condition.

## 1 Introduction

In this paper we consider the double discrete related sine-Gordon ( $\Delta\Delta RsG$ ) equation we recently introduced in [1], i.e.

$$V_{l+1,m+1} = -V_{l,m} + \frac{b - a(V_{l+1,m} + V_{l,m+1})}{c(V_{l+1,m} + V_{l,m+1}) + a}. \quad (1)$$

We investigate the integrability of the mappings derived under the periodicity condition  $V_{l+\zeta_2,m-\zeta_1} = V_{l,m}$ , with  $\zeta_1$  and  $\zeta_2$  being relatively prime.<sup>1</sup> Under this periodicity condition, we derive the  $(\zeta_1 + \zeta_2)$  dimensional map

$$V'_0 = V_1, \dots, V'_{\zeta_1+\zeta_2-2} = V_{\zeta_1+\zeta_2-1}, V'_{\zeta_1+\zeta_2-1} = -V_0 + \frac{b - a(V_{\zeta_1} + V_{\zeta_2})}{c(V_{\zeta_1} + V_{\zeta_2}) + a}, \quad (2)$$

where  $V_{l,m} = V_n$  and  $n = \zeta_1 l + \zeta_2 m$ . When the map (2) is odd dimensional, it can be reduced to an even dimensional map under the reduction  $v_i = V_i + V_{i+1}$ .

The plan of this paper is as follows : in section 2 we show that the mapping (2), when  $\zeta_1$  odd and  $\zeta_2$  even or  $\zeta_1$  even and  $\zeta_2$  odd, is related to the mapping derived from the double discrete (alternative) Korteweg-de Vries (a  $\Delta\Delta KdV$ ) equation (given in [1]) when  $\zeta_1$  odd and  $\zeta_2$  or  $\zeta_1$  even and  $\zeta_2$  odd, respectively. In section 3 we show that in the special case  $\zeta_1 = 1$  and  $\zeta_2$  even, the mapping (2) is a particular case of the *hierarchy of integrable asymmetric mappings*<sup>2</sup> we introduced in [1].

## 2 Double discrete related sine-Gordon ( $\Delta\Delta RsG$ ) equation

Consider the alternative  $\Delta\Delta KdV$  (a  $\Delta\Delta KdV$ ) equation given in [1], i.e.

$$V_{l+1,m+1} = V_{l,m} + \frac{(\epsilon - \gamma)(V_{l+1,m} - V_{l,m+1}) + \xi}{\beta(V_{l+1,m} - V_{l,m+1}) + \gamma}. \quad (3)$$

There are three possible cases to consider using the periodicity condition  $V_{l+\zeta_2,m-\zeta_1} = V_{l,m}$ , 1)  $\zeta_1$  odd and  $\zeta_2$  even, 2)  $\zeta_1$  even and  $\zeta_2$  odd, and 3)  $\zeta_1$  odd and  $\zeta_2$  odd.

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<sup>1</sup>For a detailed discussion of mappings derived from partial difference equations under such a periodicity condition see [3].

<sup>2</sup>We call a mapping, which possesses at least one cyclic invariant  $n$ -quadratic integral, *symmetric* if the cyclic invariant integral is invariant under any permutation of the variables, see [1]. We call a mapping *asymmetric* if the  $n$ -quadratic integral is not invariant under any permutation of the variables.

Case 1) When  $\gamma = \epsilon = 0$  together with

$$V_{l,m} \rightarrow \begin{cases} V_{l,m}, & l \text{ even} \\ -V_{l,m}, & l \text{ odd} \end{cases} \quad (4)$$

we obtain

$$V_{l+1,m+1} = -V_{l,m} + \frac{\xi}{\beta(V_{l+1,m} + V_{l,m+1})}. \quad (5)$$

Case 2) When  $\gamma = \epsilon = 0$  and

$$V_{l,m} \rightarrow \begin{cases} V_{l,m}, & m \text{ even} \\ -V_{l,m}, & m \text{ odd} \end{cases} \quad (6)$$

we obtain

$$V_{l+1,m+1} = -V_{l,m} - \frac{\xi}{\beta(V_{l+1,m} + V_{l,m+1})}. \quad (7)$$

In cases 1) and 2) the periodicity condition remains invariant under the transformation (4) and (6) and as a result the mapping derived from (3) (with  $\gamma = \epsilon = 0$ ) is the same as that derived from (5) or (7). In case 3), however, the periodicity condition does not remain invariant under the transformation (4) and (6) and as a result the mapping derived is different. We will not consider case 3) in this paper.

The mapping derived from (5) or (7) under the periodicity condition  $V_{l+\zeta_2, m-\zeta_1} = V_{l,m}$  is

$$V'_0 = V_1, \dots, V'_{\zeta_1+\zeta_2-2} = V_{\zeta_1+\zeta_2-1}, V'_{\zeta_1+\zeta_2-1} = -V_0 \pm \frac{\xi}{\beta(V_{\zeta_1} + V_{\zeta_2})}, \quad (8)$$

with  $+$  sign for case 1) and  $-$  sign for case 2). Using the translations  $V_i \rightarrow V_i + a/2\beta$  and  $V'_i \rightarrow V'_i + a/2\beta$  and setting  $(\beta, \xi) = (c, \pm(a^2/c + b))$ , the mapping (8) becomes the mapping (2).

We now turn to the integrals of the mapping (2). We consider case 1) only, case 2) is similar. Under the periodicity condition  $V_{l+\zeta_2, m-\zeta_1} = V_{l,m}$  and  $\gamma = \epsilon = 0$ , (3) can be used to derive the  $(\zeta_1 + \zeta_2)$  dimensional map

$$V'_0 = V_1, \dots, V'_{\zeta_1+\zeta_2-2} = V_{\zeta_1+\zeta_2-1}, V'_{\zeta_1+\zeta_2-1} = V_0 + \frac{\xi}{\beta(V_{\zeta_1} - V_{\zeta_2})}. \quad (9)$$

Under the transformations  $V_i \rightarrow (-1)^i V_i$  and  $V'_i \rightarrow (-1)^i V'_i$  the map (9) becomes

$$V'_0 = -V_1, V'_1 = -V_2, \dots, V'_{\zeta_1+\zeta_2-2} = -V_{\zeta_1+\zeta_2-1}, V'_{\zeta_1+\zeta_2-1} = V_0 - \frac{\xi}{\beta(V_{\zeta_1} + V_{\zeta_2})}. \quad (10)$$

We note that the map (10) can be written as  $L_{10} = L \circ L_8 = L_8 \circ L$ , where  $L$  is defined to be the map with coordinates  $V'_i = -V_i$  (for all  $i$ ) and  $L_8$  is the map (8). Using this, we see that  $L_{10} \circ L_{10} = L \circ L_8 \circ L \circ L_8 = L \circ L \circ L_8 \circ L_8 = L_8 \circ L_8$ , as  $L$  is an involution, i.e.  $L \circ L = \text{id}$ , where  $\text{id}$  is the identity map. This shows that for all integrals  $I_i$  of the map  $L_{10}$ , we have  $I_i(V_0, \dots, V_{\zeta_1+\zeta_2-1}) = I_i(-V'_0, \dots, -V'_{\zeta_1+\zeta_2-1}) = I_i(V''_0, \dots, V''_{\zeta_1+\zeta_2-1})$  for the map (8). In fact  $I_i(V_0, \dots, V_{\zeta_1+\zeta_2-1}) = -I_i(V'_0, \dots, V'_{\zeta_1+\zeta_2-1})$ , i.e. the actual integrals of the mapping (8) are  $I_i^2$ .

### 3 $\zeta_1 = 1$ and $\zeta_2$ even

In this section we show that in the special case  $\zeta_1 = 1$  and  $\zeta_2$  even, the mapping (2) is a particular case of the hierarchy of integrable asymmetric mappings mentioned above. We illustrate this with the four-dimensional case, i.e.  $\zeta_1 = 1$  and  $\zeta_2 = 4$ .

When  $\zeta_1 = 1$  and  $\zeta_2 = 4$ , the mapping (2) becomes

$$V'_0 = V_1, V'_1 = V_2, V'_2 = V_3, V'_3 = V_4, V'_4 = -V_0 + \frac{b - a(V_1 + V_4)}{c(V_1 + V_4) + a} \quad (11)$$

Under the reduction  $w = V_0 + V_1$ ,  $x = V_1 + V_2$ ,  $y = V_2 + V_3$  and  $z = V_3 + V_4$ , we obtain the four-dimensional mapping

$$w' = x, \quad x' = y, \quad y' = z, \quad z' = -w + \frac{c(x - y + z)^2 + b}{c(x - y + z) + a}. \quad (12)$$

Consider the four-dimensional case of the hierarchy of integrable asymmetric mappings given in [1], i.e. the mapping

$$\begin{aligned} w' &= -w - \frac{\beta(x + y + z)^2 + \epsilon(x + y + z) + \xi_0}{\beta(x + y + z) + \gamma_0} \\ x' &= -x - \frac{\beta(w' + y + z)^2 + \epsilon(w' + y + z) + \xi_1}{\beta(w' + y + z) + \gamma_1} \\ y' &= -y - \frac{\beta(w' + x' + z)^2 + \epsilon(w' + x' + z) + \xi_2}{\beta(w' + x' + z) + \gamma_2} \\ z' &= -z - \frac{\beta(w' + x' + y')^2 + \epsilon(w' + x' + y') + \xi_3}{\beta(w' + x' + y') + \gamma_3}, \end{aligned} \quad (13)$$

Under the substitutions  $(w, x, y, z) \rightarrow (w, -x, y, -z)$  and  $(w', x', y', z') \rightarrow (w', -x', y', -z')$  the mapping (13) becomes

$$\begin{aligned} w' &= -w + \frac{\beta(x - y + z)^2 - \epsilon(x - y + z) + \xi_0}{\beta(x - y + z) - \gamma_0} \\ x' &= -x + \frac{\beta(w' + y - z)^2 + \epsilon(w' + y - z) + \xi_1}{\beta(w' + y - z) + \gamma_1} \\ y' &= -y + \frac{\beta(-w' + x' + z)^2 - \epsilon(-w' + x' + z) + \xi_2}{\beta(-w' + x' + z) - \gamma_2} \\ z' &= -z + \frac{\beta(w' - x' + y')^2 + \epsilon(w' - x' + y') + \xi_3}{\beta(w' - x' + y') + \gamma_3}, \end{aligned} \quad (14)$$

If we set  $\xi_0 = \xi_1 = \xi_2 = \xi_3 = \xi$ ,  $\gamma_0 = -\gamma_1 = \gamma_2 = -\gamma_3 = -\gamma$  and  $\epsilon = 0$  the mapping (14) can be written as  $L_1^4 = L_1 \circ L_1 \circ L_1 \circ L_1$ , where  $L_1$  is given by

$$\tilde{w} = x, \quad \tilde{x} = y, \quad \tilde{y} = z, \quad \tilde{z} = -w + \frac{\beta(x - y + z)^2 + \xi}{\beta(x - y + z) + \gamma}, \quad (15)$$

which is equivalent to (12). The integrals,  $I_1$  and  $I_2$  of the mapping (15), can be obtained from the coefficients of the different powers of  $\lambda$  of the characteristic equation

$$C(h, \lambda) = \det(\lambda I - L(h)) = 0, \quad (16)$$

where  $I$  is the identity matrix and  $L(h)$  is the  $L$  matrix of the Lax pair  $(L, M)$  for the mapping (13), see [1]. We note that the integrals,  $I_1$  and  $I_2$ , satisfy  $I_i(w, x, y, z) = -I_i(w', x', y', z')$ , for  $i = 1, 2$ , i.e. the actual integrals for the mapping (15) are  $\bar{I}_i = I_i^2$ .<sup>3</sup> The above discussion also shows that (15) has an asymmetric form, i.e. (14), which is equivalent to (13).

This procedure can be applied to any even  $\zeta_2$  (with  $\zeta_1 = 1$ ). Showing that the mappings obtained from the  $(\Delta\Delta RsG)$ , with  $\zeta_1 = 1$  and  $\zeta_2$  even, are special cases of the hierarchy of integrable asymmetric mappings introduced in [1].

## 4 Conclusion

In this paper we have shown that the mappings derived from the  $(\Delta\Delta RsG)$  equation, when  $\zeta_1$  odd and  $\zeta_2$  even or  $\zeta_1$  even and  $\zeta_2$  odd, are related to the mappings derived from the  $a \Delta\Delta KdV$  equation,

<sup>3</sup>The symplectic structure for the mapping (15) is given in [1]. Using this symplectic structure, one can easily show that the two functionally independent integrals,  $\bar{I}_1$  and  $\bar{I}_2$ , are in involution.

when  $\zeta_1$  odd and  $\zeta_2$  or  $\zeta_1$  even and  $\zeta_2$  odd, respectively. We have also shown that the mappings derived from the  $(\Delta\Delta RsG)$  equation, when  $\zeta_1 = 1$  and  $\zeta_2$  even, are special cases of the hierarchy of integrable asymmetric mappings given in [1]. We note that a similar relationship exists between the asymmetrized mappings<sup>4</sup> obtained from the  $a \Delta\Delta MKdV$  and  $a \Delta\Delta sG$  (when  $\zeta_1 = 1$  and  $\zeta_2$  even) equations, see [1, Aappendix C]. An open question is whether a transformation exists relating these two hierarchies? If a transformation relating these two hierarchies does exist, then the hierarchy of asymmetric mappings given in [1, Aappendix C] is integrable (which we expect). Another open question is whether the mappings derived from the  $a \Delta\Delta KdV$ ,  $a \Delta\Delta MKdV^5$ ,  $a \Delta\Delta sG^5$  and  $\Delta\Delta RsG$  equations under an appropriate periodicity condition are special cases of the hierarchy of integrable asymmetric mappings given in [1].

## Appendix A. Intermediate forms

In [1] we showed how to determine intermediate forms (i.e. where some symmetry in the variables exists) that are integrable using nonautonomous mappings. In this appendix we show another way of obtaining such forms, but this time using the hierarchy of autonomous integrable asymmetric mappings. We illustrate with the four dimensional case.

Consider (13) when  $\xi_2 = \xi_0$ ,  $\xi_3 = \xi_1$ ,  $\gamma_2 = \gamma_0$  and  $\gamma_3 = \gamma_1$ , i.e.

$$\begin{aligned} w' &= -w - \frac{\beta(x+y+z)^2 + \epsilon(x+y+z) + \xi_0}{\beta(x+y+z) + \gamma_0} \\ x' &= -x - \frac{\beta(w'+y+z)^2 + \epsilon(w'+y+z) + \xi_1}{\beta(w'+y+z) + \gamma_1} \\ y' &= -y - \frac{\beta(w'+x'+z)^2 + \epsilon(w'+x'+z) + \xi_0}{\beta(w'+x'+z) + \gamma_0} \\ z' &= -z - \frac{\beta(w'+x'+y')^2 + \epsilon(w'+x'+y') + \xi_1}{\beta(w'+x'+y') + \gamma_1}. \end{aligned} \quad (17)$$

The mapping (17) can be written as  $L = L_2^2 = L_2 \circ L_2$ , where  $L_2$  is given by

$$\begin{aligned} \hat{w} &= y \\ \hat{x} &= z \\ \hat{y} &= -w - \frac{\beta(x+y+z)^2 + \epsilon(x+y+z) + \xi_0}{\beta(x+y+z) + \gamma_0} \\ \hat{z} &= -x - \frac{\beta(y+\hat{y}+z)^2 + \epsilon(y+\hat{y}+z) + \xi_1}{\beta(y+\hat{y}+z) + \gamma_1}. \end{aligned} \quad (18)$$

This mapping is equivalent to the mapping (D.4) of [1]. In fact, if we make the substitutions  $(w, x, y, z) \rightarrow (w, y, x, z)$  and  $(\hat{w}, \hat{x}, \hat{y}, \hat{z}) \rightarrow (\hat{w}, \hat{y}, \hat{x}, \hat{z})$  in the mapping (18) we obtain the mapping (D.4). The integrals of (D.4) can be obtained in the same way, noting that the integrals of (18) can be obtained from the Lax pair given in [1].

More generally, the intermediate forms can be determined in the following way : find the possible integers  $i$  that divide  $n$ , e.g for  $n = 16$  we have  $i = 1, 2, 4, 8, 16$ . Set the  $\gamma_k$ 's and  $\xi_k$ 's to  $\gamma_l$  and  $\xi_l$  ( $l = 0, \dots, i-1$ ) in such a way that we have  $i$  of each<sup>6</sup> and the resulting asymmetric map can be written as  $L = L_i^{n/i}$ . The intermediate form is  $L_i$ .<sup>7</sup>

The advantages in this method are 1) the mappings obtained need not be in general complex, whereas the mappings obtained using the method given in [1] are, and 2) the intermediate forms are easily obtained.

<sup>4</sup>There are two minor errors in [1, appendix C] : 1) the transformation reducing the asymmetric  $(2m+1)$ -dimensional mappings to  $(2m)$ -dimensional mappings is actually  $V_i = v_i v_{i+1}$ , and 2) two lines below the previous error,  $v_i = V_i V_{i+1}$ .

<sup>5</sup>See [1] for the prefix  $a$ .

<sup>6</sup>This is achieved by setting the first  $i$   $\gamma_k$ 's and  $\xi_k$ 's ( $k = 0, \dots, i-1$ ) to  $(\gamma_0, \dots, \gamma_{i-1})$  and  $(\xi_0, \dots, \xi_{i-1})$ , respectively, and then repeating the process another  $i-1$  times for the remaining  $\gamma_k$ 's and  $\xi_k$ 's, see example above.

<sup>7</sup>This way of constructing intermediate forms can also be applied to the asymmetric mappings given in [1, appendix C].

## Appendix B. Involutions

In this appendix we show that all the mappings given in this paper, and the mappings given in [1] and [2], can be written as a product (composition) of involutions.

First, we note that the cyclic shift,  $L_c$ , i.e.

$$v'_0 = v_1, \dots, v'_{n-1} = v_n, v'_n = v_0 \quad (19)$$

can be written as a product of involutions, i.e.  $L_c = L_{12} \circ L_{01} \circ L_{03} \circ L_{04} \circ \dots \circ L_{0n}$  (for  $n \geq 3$ )<sup>8</sup>, where the involution  $L_{ij}$  is defined as  $v'_k = v_k$  ( $k \neq i, j$ ) together with  $v'_i = v_j$  and  $v'_j = v_i$  ( $i < j$ ).

The mapping (15) can be written as  $L_1 = L_z \circ L_c = L_z \circ L_{12} \circ L_{01} \circ L_{03}$ , where

$$w' = w, \quad x' = x, \quad y' = y, \quad z' = -z + \frac{\beta(w - x + y)^2 + \xi}{\beta(w - x + y) + \gamma} \quad (20)$$

and  $w = v_0, x = v_1, y = v_2$  and  $z = v_3$ .

The mapping (17) can be written as  $L_2 = L_z \circ L_y \circ L_{02} \circ L_{13}$ , where  $L_z$  is defined as

$$w' = w, \quad x' = x, \quad y' = y, \quad z' = -z - \frac{\beta(w + x + y)^2 + \epsilon(w + x + y) + \xi_1}{\beta(w + x + y) + \gamma_1}, \quad (21)$$

$L_y$  is defined as

$$w' = w, \quad x' = x, \quad y' = -y - \frac{\beta(w + x + z)^2 + \epsilon(w + x + y) + \xi_0}{\beta(w + x + z) + \gamma_0}, \quad z' = z \quad (22)$$

and  $w = v_0, x = v_1, y = v_2$  and  $z = v_3$ .

The mappings given in the papers [1] and [2] can be written as a product of involutions or as an involution composed with a cyclic shift. For the latter, the above discussion shows that these mappings can be written as a product of involutions. Finally, for the mapping given in [2], i.e.

$$x' = y, \quad y' = z, \quad z' = x + \frac{A(y - z)}{Byz + C} \quad (23)$$

we have  $L_3 = L_z \circ L_c \circ L_- = L_z \circ L_{12} \circ L_{01} \circ L_-$ , where  $L_z$  is given by

$$x' = x, \quad y' = y, \quad z' = -z + \frac{A(x - y)}{Bxy + C}, \quad (24)$$

$L_-$  is given by

$$x' = -x, \quad y' = y, \quad z' = z \quad (25)$$

and  $x = v_0, y = v_1$  and  $z = v_2$ . We note that  $L_-$  is an involution.

All of the above mappings are invertible since they are a product of involutions, i.e.  $L = L_0 \circ L_1 \circ \dots \circ L_{m-1} \circ L_m$  (where the  $L_i$  are involutions). Their inverse is  $L^{-1} = L_m \circ L_{m-1} \circ \dots \circ L_1 \circ L_0$ .

## References

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<sup>8</sup>The three-dimensional case is  $L_c = L_{12} \circ L_{01}$ .